LOCAL ENERGY DECAY FOR THE WAVE EQUATION WITH A NONLINEAR TIME DEPENDENT DAMPING.

AHMED BCHATNIA AND MOEZ DAOULATLI

Abstract. This paper addresses a wave equation on a exterior domain in $\mathbb{R}^d$ (d odd) with nonlinear time dependent dissipation. Under a microlocal geometric condition we prove that the decay rates of the local energy functional are obtained by solving a nonlinear non-autonomous differential equation.

1. Introduction and Statement of the result

Let O be a compact domain of $\mathbb{R}^d$ ($d \geq 3$ is odd) with C^∞ boundary $\Gamma = \partial\Omega$ and $\Omega = \mathbb{R}^d \setminus O$, $O \subset B_R$ for some $R > 0$. Consider the following wave equation with a nonlinear time dependent damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\rho(t) g(\partial_t u) = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0, & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(0,x) = \varphi_0 \quad \text{and} \quad \partial_t u(0,x) = \varphi_1. & \end{cases} \tag{1.1}$$

Here Δ denotes the Laplace operator in the space variables.

The nonlinear terms satisfy:

- $a(x)$ is a non-negative function in $C^\infty(\Omega)$ with compact support such that $\operatorname{supp} a \subset B_R$.
- ρ is a positive, monotone and differentiable function on $\mathbb{R}_+$: there exists a positive constant $C_0 > 0$ such that
$$|\rho'(t)| \leq C_0 \rho(t), \text{ for all } t \geq 0.$$
Moreover, without loss of generality we assume that $\rho(0) = 1$.
- $g : \mathbb{R} \to \mathbb{R}$ is a continuous and monotone increasing function with $g(0) = 0$.

The natural space of initial data is $H = H_D(\Omega) \times L^2(\Omega)$ which is the completion of $(C_0^\infty(\Omega))^2$ with respect to the norm

$$\|\varphi\|_H^2 = \|(\varphi_0, \varphi_1)\|_H^2 = \frac{1}{2}\int_\Omega |\nabla\varphi_0|^2 + |\varphi_1|^2 \, dx.$$

It is known that (see Lions–Strauss [16]) under the conditions above, the system (1.1) is well posed in the space H, i.e., for any initial state $(u_0, u_1) \in H$ there exists a unique

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weak solution of (1.1) such that

$$u \in C^0\left(\mathbb{R}_+, H_D(\Omega)\right); \ \partial_t u \in C^0\left(\mathbb{R}_+, L^2(\Omega)\right).$$

For every $t \in \mathbb{R}_+$, we define the evolution operator $U(t)$ by

$$\begin{array}{rcccl} U(t): & H & \longrightarrow & H & \\ & (u_0, u_1) & \longmapsto & U(t)(u_0, u_1) & = (u(t), \partial_t u(t)), \end{array}$$

where u is the solution of (1.1). Let us consider the energy at instant t defined by

$$\begin{aligned} E_u(t) &= \frac{1}{2}\int_\Omega \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx \\ &= \|U(t)\varphi\|_H^2 . \end{aligned}$$

We formally obtain the following identity

$$E_u(T) + \int_0^T \int_\Omega a(x)\, g(\partial_t u)\, \partial_t u \ dx \ dt = E_u(0), \tag{1.2}$$

for every $T \geq 0$. We define the local energy by

$$\begin{aligned} E_r(u)(t) &= \frac{1}{2}\int_{\Omega\cap B_r} \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx \\ &= \|(u(t), \partial_t u(t))\|_{H(B_r)}^2 \end{aligned}$$

where $B_r = \left\{x \in \mathbb{R}^d, |x| < r\right\}$, contains the obstacle O.

Our goal is to give the rate of decay of the local energy. Throughout the paper we will frequently invoke the following notation

$$\Omega_{s,t} = [s,t] \times \Omega, \ t \geq s \geq 0 \text{ and } \Omega_{0,t} = \Omega_t.$$

For convenience we also introduce the following weighted measure on Ω:

$$\mathfrak{m}_a = a(x)\, dxdt.$$

The problem of the local energy decay for the wave equation and systems on exterior domain has been intensively investigated during the last decades. For the classical wave equation, the story goes up to the pioneering works of Lax-Phillips [14], and Morawetz, Strauss and Ralston [20]. When the obstacle is trapping, Ralston [23] proved that there is no uniform decay rate, and Morawetz-Ralston-Strauss [20] and Melrose [18] obtained the exponential decay for nontrapping obstacle. In [5], without any assumption on the dynamics Burq proved the logarithmic decay of the local energy with respect to any Sobolev norm larger than the initial energy. Nakao in [22] proved that the local energy decay exponentially if d is odd and polynomially if d is even under the Lions's geometric condition. Later, in [1] and for general obstacles, Aloui and Khenissi proved the exponential decay of the local energy by mean of linear internal localized damping and Khenissi [11] proved the polynomial decay in even space dimension. For that, they introduced the exterior geometric control condition (E.G.C.) inspired from the so-called microlocal condition of Bardos-Lebeau-Rauch [2] and they used in a crucial way the propagation properties of the microlocal defect measures of Gérard [10] (see also Lebeau [15]). More recently in [6], using a nonlinear internal localized damping,

Daoulatli obtained various decay rates depending on the behavior of the damping term. Concerning the semilinear waves on unbounded domains, we quote the work of Bchatnia and Daoulatli [3], which establishes an exponential decay of the local energy for the solutions of subcritical wave equation outside convex obstacle. We also mention the result of Daoulatli et al [8] on the energy decay rates of the local energy for the elastic system with a nonlinear damping. Finally, we quote the following results on the energy decay rates for the wave equation with time dependent damping in bounded domain [4, 7, 17, 19, 21]. However, concerning the decay property of the local energy of wave equation with nonlinear time dependent dissipation in exterior domains no results seem to be known.

Now, we recall the exterior geometric condition of [1].

Definition 1.1 (EGC)**.** *Let $R>0$ such that $O\subset B_R$, $T_R>0$ and $\omega=\{x\in\Omega; a(x)>0\}$. We say that (ω,T_R) verifies the exterior geometric control condition on B_R (E.G.C), if every geodesic γ starting from B_R at time $t=0$, is such that*
$\cdot$ γ leaves $\mathbb{R}_+\times B_R$ before the time T_R, or
$\cdot$ γ meets $\mathbb{R}_+\times\omega$ between the times 0 and T_R.

In this paper, under the condition (EGC), we give the rate of decay of the local energy of solutions of the wave equation with nonlinear time dependent dissipation in exterior domain. More precisely, the rate of decay will be determined from the following nonlinear, non-autonomous ODE:

$$\frac{dS}{dt}+q\left(t,S\left(t\right)\right)=0,\qquad S\left(0\right)=E_u\left(0\right),$$

where the function q is defined in Subsection 1.2 below.

1.1. **Behavior of the dissipation at the origin and infinity.** In order to characterize decay rates for the energy, we need to introduce several special functions, which in turn will depend on the growth of g near the origin and near infinity. For that purpose let $m_0\geq 1$, and following [13], we classify the behavior of g *near the Origin*:

AO1: Linearly bounded on I:

$$\frac{1}{m_0}y^2\leq g\left(y\right)y\leq m_0y^2,\quad y\in I,$$

AO2: Superlinear on I:

$$g\left(y\right)y\leq m_0y^2,\quad y\in I,$$

AO3: Sublinear on I:

$$\frac{1}{m_0}y^2\leq g\left(y\right)y,\quad y\in I,$$

where $I=[-\eta,\eta]$ and $0<\eta<1$.

We then define a concave function h_0 which will describe the growth of g near the origin. Since g is a continuous monotone increasing function vanishing at zero, following

[12] there exists a concave monotone increasing function h_0 defined on $\mathbb{R}_+$ such that $h_0(0) = 0$ and

$$h_0\left(g\left(y\right)y\right) \geq \epsilon_0 \left(g\left(y\right)^2 + y^2\right) \quad \text{ for } \quad |y| < \eta_0, \tag{1.3}$$

for some ϵ_0, $\eta_0 > 0$. For example when g is superlinear and odd, then $h_0^{-1}(s) = \sqrt{s}g\left(\sqrt{s}\right)$ when $|s| \leq \eta$. For further details on the construction of such function we refer the interested reader to [12, 7, 9].

We now assume that g is linearly bounded at infinity:

$$\frac{1}{m}y^2 \leq g\left(y\right)y \leq my^2, \quad |y| \geq \eta_0. \tag{1.4}$$

We introduce some auxiliary functions $\alpha(t)$ and $\beta(t)$ which are linked to the function ρ as follows:

$$\beta\left(t\right) = \left\{ \begin{array}{lll} \frac{1}{T}\rho\left(t+T\right) & & \text{if } \rho \text{ is decreasing} \\ \frac{1}{T} & t < T & \multirow{2}{*}{$\left.\right\}$ if ρ is increasing} \\ \frac{1}{T}\rho\left(t-T\right) & t \geq T & \end{array} \right.$$

and

$$\alpha\left(t\right) = \begin{cases} 1 & \text{if } \rho \text{ is decreasing} \\ \rho^{-2}\left(t+T\right) & \text{if } \rho \text{ is increasing,} \end{cases}$$

here T is a positive constant which will be precised in the statement of Theorem 1.

1.2. **Auxiliary functions.** Let h_0, α and β be as defined above and set

$$h = I + \mathfrak{m}_a\left(\Omega_T\right)h_0 \circ \tfrac{I}{T\mathfrak{m}_a(\Omega_T)}. \tag{1.5}$$

We introduce, for $t \geq 0$,

$$q\left(t,.\right) = \beta\left(t\right)h^{-1} \circ \tfrac{\alpha(t)}{K}I, \tag{1.6}$$

where K is a positive constant such that $K \geq C_T$, here C_T is the constant that appears in the estimate (5.16). We note that C_T is independent of the initial data.

2. The main result

In this section we give the main result of this paper.

Theorem 1. *Let T_R be such that $\left(\{x \in \Omega; a\left(x\right) > 0\}, T_R\right)$ satisfies the exterior geometric condition on B_R. Then there exist $T \geq T_R + 9R$ and a positive constant C_T such that inequality*

$$E_R\left(u\right)\left(t\right) \leq S\left(t-T\right), \qquad \text{for all } t \geq T, \tag{2.1}$$

holds for every solution u of system (1.1) if the initial data (u_0, u_1) in the energy space H is compactly supported in B_R. Here $S(t)$ is the solution of the following nonlinear differential equation:

$$\frac{dS}{dt} + q\left(t, S\left(t\right)\right) = 0, \qquad S\left(0\right) = E_u\left(0\right), \tag{2.2}$$

and the function q *is defined in Subsection 1.2 above. Moreover, if for some* $T_0 \gg 1$

$$\int_{T_0}^{t} q(s,\gamma)\, ds \underset{t\to+\infty}{\longrightarrow} \infty, \tag{2.3}$$

for every $0 < \gamma \ll 1$, *then*

$$E_R(u)(t) \underset{t\to+\infty}{\longrightarrow} 0.$$

The proof of theorem will be stated in Section 5. In the next section we give some applications of this result.

3. Applications

We recall first the following lemma useful to us to determine the rate of decay.

Lemma 3.1 ([7])**.** (1) *Let* α_1 *is a positive, differentiable and decreasing function on* $\mathbb{R}_+$ *and* β_1 *is a non-negative function on* $\mathbb{R}_+$. *Let* S *be a positive function verifying the following differential inequality*

$$\frac{dS}{dt} + \beta_1(t)\, p(\alpha_1(t) S) \le 0, \; S(0) > 0. \tag{3.1}$$

We assume that p *is a strictly increasing function on* $[0, \alpha_1(0) S(0)]$ *with* $p(0) = 0$ *and verifies*

$$p(x) \le m^{-1} x \quad \text{for all } x \le \alpha_1(0) S(0) \;\text{ and for some } m > 0.$$

Then we have

$$S(t) \le \frac{1}{\alpha_1(t)} \psi^{-1}\left(\int_0^t \alpha_1(s)\beta_1(s)\, ds - m \ln\left(\frac{\alpha_1(t)}{\alpha_1(0)}\right)\right), \forall t \ge 0, \tag{3.2}$$

where

$$\psi(x) = \int_x^{\alpha_1(0)S(0)} \frac{ds}{p(s)}.$$

(2) *If in addition the function* p *satisfies the following property*

$$p(\alpha_1(t)x) \ge m p(x)\, p(\alpha_1(t)), \; \forall t \ge 0, \; \text{and } x \in [0, S(0)] \tag{3.3}$$

for some $m > 0$ *and* p *is a strictly increasing function on* $[0, S(0)]$, *then the function* S *of (3.1), verifies*

$$S(t) \le \psi^{-1}\left(\int_0^t m p(\alpha_1(s))\, \beta_1(s)\, ds\right), \forall t \ge 0$$

where

$$\psi(x) = \int_x^{S(0)} \frac{ds}{p(s)}.$$

Remark 3.1. *The rate of decay of the energy depends on* α, β *and the behavior of* h^{-1} *near zero. To determine it, we only have to find* $0 < \epsilon_0 \leq 1$, *such that*

$$\chi(s) \leq h^{-1}(s), \text{ for every } s \leq \epsilon_0,$$

with

$$\chi(s) = C_1 h_0^{-1}\left(\tfrac{s}{2C_2}\right), \text{ for every } s \leq \epsilon_0,$$

where

$$C_1 = \min\left(T\mathfrak{m}_a(\Omega), 1\right) \text{ and } C_2 = \max\left(T\mathfrak{m}_a(\Omega), 1\right).$$

In the sequel c denotes a positive constant which is independent of the initial data. Moreover, if ρ is increasing, we suppose that there exist t_0, $c_0 > 0$, such that

$$\rho(t - 2T) \geq c_0 \rho(t) \quad \text{for every } t \geq t_0. \tag{3.4}$$

3.1. **The linear case.** Let $g(s) = s$, $0 \leq |s| < 1$. According to (1.3), the auxiliary function h is defined as $h(y) = 2y$. Then

$$\psi(x) = 2\ln\left(\frac{\|\varphi\|_H^2}{x}\right)$$

(1) ρ is decreasing: $\alpha_1(t) = \frac{1}{K}$ and $\beta_1(t) = \frac{1}{T}\rho(t+T)$. As a result, using Theorem 1 and some computations, the estimate

$$E_R(u)(t) \leq S(t-T), \qquad t \geq T$$

becomes

$$E_R(u)(t) \leq c\|\varphi\|_H^2 \exp\left(-\tfrac{1}{KT}\int_0^t \rho(s)\,ds\right), \text{ for all } t \geq 0.$$

(2) ρ is increasing: $\alpha_1(t) = \frac{1}{K\rho^2(t+T)}$ and $\beta_1(t) = \frac{1}{T}\rho(t-T)$, $t \geq T$. Then

$$E_R(u)(t) \leq c\|\varphi\|_H^2 \exp\left(-\tfrac{1}{KT}\int_{2T}^t \tfrac{\rho(s-2T)}{\rho^2(s)}ds\right), \text{for all } t \geq 2T.$$

Now, using (3.4) and making some arrangement, we obtain

$$E_R(u)(t) \leq c\|\varphi\|_H^2 \exp\left(-\tfrac{c_0}{KT}\int_0^t \rho^{-1}(s)\,ds\right), \text{ for all } t \geq 0.$$

Remark 3.2. *An important special case of (1.1) is when* $\rho(t) = (1+t)^\tau$, $\tau \in \mathbb{R}$. *We have*

$$\begin{aligned}
&E_R(u)(t) \leq cE_u(0)\exp\left(-c_K(1+t)^{1-|\tau|}\right) && |\tau| < 1,\\
&E_R(u)(t) \leq cE_u(0)(1+t)^{-\frac{1}{KT}} && \tau = -1,\\
&E_R(u)(t) \leq cE_u(0)(1+t)^{-\frac{c_0}{KT}} && \tau = 1,
\end{aligned}$$

for every $t \geq 0$. *Moreover, it is clear that, we cannot obtain the decay to zero of the energy when* $|\tau| > 1$.

3.2. **The nonlinear case.** Since g is linearly bounded near infinity, there exists a constant $A>0$, such that

$$T\mathfrak{m}_a(\Omega)\, h_0^{-1}\left(\tfrac{s}{2T\mathfrak{m}_a(\Omega)}\right)\le h^{-1}(s)\le s, \text{ for every } s\le A. \tag{3.5}$$

So the rate of decay of the energy depends only on the behavior of g near the origin.

Example 3.1 (Superlinear polynomial damping near the origin)**.** *Suppose* $g(s)=s|s|^{r_0-1}$, $0\le |s|<1$ *for some* $r_0>1$. *The auxiliary function* h_0 *which may be defined as*

$$h_0^{-1}(s)=s^{\frac{1+r_0}{2}},\ \text{for } s\in[0,1].$$

Consequently, we obtain

$$\psi(x)\le \tfrac{2}{r_0-1}\left(x^{\frac{1-r_0}{2}}-\|\varphi\|_H^{1-r_0}\right).$$

(1) ρ *decreasing. Let* $\alpha_1(t)=(2TK)^{-1}$ *and* $\beta_1(t)=\rho((t+1)T)$. *Then we deduce that*

$$E_R(u)(t)\le\|\varphi\|_H^2\left(1+\left(\tfrac{\|\varphi\|_H^2}{2TK}\right)^{\frac{r_0-1}{2}}\tfrac{r_0-1}{4KT}\int_T^t\rho(s)\,ds\right)^{-\frac{2}{r_0-1}};\qquad t\ge T.$$

After some arrangement and by choosing K *big enough, we obtain*

$$E_R(u)(t)\le c\|\varphi\|_H^2\left(1+\left(\tfrac{\|\varphi\|_H^2}{K}\right)^{\frac{r_0-1}{2}}\tfrac{c}{KT}\int_0^t\rho(s)\,ds\right)^{-\frac{2}{r_0-1}},\ t\ge 0.$$

(2) ρ *increasing. Let* $\alpha_1(t)=\left(2TK\rho^2(t+T)\right)^{-1}$ *and* $\beta_1(t)=\rho(t-T)$. *Then,*

$$E_R(u)(t)\le c\|\varphi\|_H^2\left(1+\left(\tfrac{\|\varphi\|_H^2}{K}\right)^{\frac{r_0-1}{2}}\tfrac{c}{KT}\int_{2T}^t\tfrac{\rho(s-2T)}{(\rho(s))^{r_0+1}}ds\right)^{-\frac{2}{r_0-1}},$$

for $t\ge 2T$.

Using (3.4) and after some computation, we obtain

$$E_R(u)(t)\le c\|\varphi\|_H^2\left(1+\left(\tfrac{\|\varphi\|_H^2}{K}\right)^{\frac{r_0-1}{2}}\tfrac{c}{KT}\int_0^t(\rho(s))^{-r_0}\,ds\right)^{-\frac{2}{r_0-1}},\ t\ge 0.$$

Remark 3.3. *Take* $\rho(t)=(1+t)^\tau$, $\tau\in\left[-1,\frac{1}{r_0}\right]$. *We have*

$$E_R(u)(t)\le C_K(\ln(2+t))^{-\frac{2}{r_0-1}},\quad \tau=-1 \text{ or } \tau=\tfrac{1}{r_0},$$
$$E_R(u)(t)\le C_K(1+t)^{\mu},\qquad \tau\in\left]-1,\tfrac{1}{r_0}\right[,$$

for every $t\ge0$, *with*

$$\mu=-\tfrac{2}{r_0-1}(1+\tau),\qquad -1<\tau\le 0,$$
$$\mu=-\tfrac{2}{r_0-1}(1-\tau r_0),\quad 0\le\tau<\tfrac{1}{r_0}.$$

Example 3.2 (Sublinear near the origin). *Assume* $g(s) = s|s|^{\theta_0 - 1}$, $0 \leq |s| < 1$, $\theta_0 \in (0,1)$. *We have*

$$h_0^{-1}(s) = s^{\frac{1+\theta_0}{2\theta_0}}, \text{ for } s \in [0,1].$$

Consequently, we infer that

$$\psi(x) \leq \tfrac{2\theta_0}{1-\theta_0}\left(x^{-\frac{1-\theta_0}{2\theta_0}} - \|\varphi\|_H^{-\frac{1-\theta_0}{\theta_0}}\right).$$

(1) *ρ decreasing.* $\alpha_1(t) = (2TK)^{-1}$ *and* $\beta_1(t) = \rho(t+T)$. *Similarly as in example 1, we obtain*

$$E_R(u)(t) \leq c\|\varphi\|_H^2\left(1 + \left(\tfrac{\|\varphi\|_H^2}{K}\right)^{(1-\theta_0)/2\theta_0}\tfrac{c}{KT}\int_0^t \rho(s)\,ds\right)^{-2\theta_0/(1-\theta_0)}, \ t \geq 0.$$

(2) *ρ increasing.* $\alpha_1(t) = \left(2TK\rho^2((t+1)T)\right)^{-1}$ *and* $\beta_1(t) = \rho(t-T)$. *Then, for all* $t \geq 0$

$$E_R(u)(t) \leq c\|\varphi\|_H^2\left(1 + \left(\tfrac{\|\varphi\|_H^2}{K}\right)^{(1-\theta_0)/2\theta_0}\tfrac{c}{KT}\int_0^t (\rho(s))^{-1/\theta_0}\,ds\right)^{-2\theta_0/(1-\theta_0)}.$$

Remark 3.4. *Take* $\rho(t) = (1+t)^\tau$, $\tau \in [-1,\theta_0]$. *We have*

$$\begin{aligned} &E_R(u)(t) \leq C_K(\ln(2+t))^{-2\theta_0/(1-\theta_0)}, && \tau = -1 \text{ or } \tau = \theta_0, \\ &E_R(u)(t) \leq C_K(1+t)^\mu, && \tau \in\,]-1,\theta_0[, \end{aligned}$$

for every $t \geq 0$, *with*

$$\begin{aligned} &\mu = -\tfrac{2\theta_0}{1-\theta_0}(1+\tau), && -1 < \tau \leq 0, \\ &\mu = -\tfrac{2\theta_0}{1-\theta_0}\left(1 - \tfrac{\tau}{\theta_0}\right), && 0 \leq \tau < \theta_0. \end{aligned}$$

Example 3.3 (Exponential damping at the origin). $g(s) = se^{-1/s^2}$, $0 < |s| < 1$. *We take*

$$h_0^{-1}(s) = se^{-1/s}, \text{ for } s \in [0,1] \tag{3.6}$$

We assume that ρ is decreasing. Let $\alpha_1(t) = (2TK)^{-1}$ *and* $\beta_1(t) = \rho(t+T)$. *Then*

$$\psi(x) \leq \|\varphi\|_H^2\left(\exp\left(\frac{1}{x}\right) - \exp\left(\frac{2TK}{\|\varphi\|_H^2}\right)\right).$$

Consequently, we find

$$E_R(u)(t) \leq 2TK\left[\ln\left(\tfrac{1}{2KT\|\varphi\|_H^2}\int_T^t \rho(s)\,ds + \exp\left(\tfrac{2TK}{\|\varphi\|_H^2}\right)\right)\right]^{-1}, \quad t \geq T,$$

and we deduce that

$$E_R(u)(t) \leq cK\left(\ln\left(\tfrac{1}{2KT\|\varphi\|_H^2}\int_0^t \rho(s)\,ds + 2\right)\right)^{-1}, \qquad t \geq 0.$$

Remark 3.5. *Take $\rho(t) = (1+t)^{\tau}$, $\tau \in [-1,0]$. We have*

$$E_R(u)(t) \leq C_K \left(\ln\left(\frac{\ln(1+t)}{KT}+2\right)\right)^{-1}, \quad \tau = -1,$$
$$E_R(u)(t) \leq C_K \left(\ln\left(\frac{1+t}{KT}+2\right)\right)^{-1}, \qquad \tau \in \left]-1,0\right],$$

for every $t \geq 0$.

4. Lax-Phillips theory and preliminary results

This section is devoted to some results on the Lax-Phillips Theory [14], which are useful for the definition and the essential properties of the Lax-Phillips semi-group.

Let us consider the free wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \mathbb{R}\times\mathbb{R}^d, \\ u(0,.) = \varphi_0 \ , \ \partial_t u(0,.) = \varphi_1. \end{cases} \tag{4.1}$$

We recall that the solution of (4.1) is given by the propagator

$$U_0(t) : H_0 \ni \varphi = (\varphi_0,\varphi_1) \to U_0(t)\varphi = (u, \partial_t u) \in H_0. \tag{4.2}$$

where H_0 is the completion of $(C_0^\infty(\mathbb{R}^d))^2$ with respect to the norm

$$\|\varphi\|_{H_0}^2 = \frac{1}{2}\int_{\mathbb{R}^d} (|\nabla\varphi_1|^2 + |\varphi_2|^2)dx.$$

Following Lax and Phillips [14], we denote :

$$D_+^0 = \{\varphi = (\varphi_0,\varphi_1) \in H_0 \ ; U_0(t)\varphi = 0 \text{ on } |x| < t \ , \ t \geq 0\},$$

the space of outgoing data, and

$$D_-^0 = \{\varphi = (\varphi_0,\varphi_1) \in H_0 \ ; U_0(t)\varphi = 0 \text{ on } |x| < -t \ , \ t \leq 0\},$$

the space of incoming data associated to the solutions of (4.1). We consider the wave equation in the exterior domain Ω.

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \mathbb{R}_+\times\Omega, \\ u = 0 & \mathbb{R}_+\times\Gamma, \\ u(0,x) = \varphi_0 \text{ and } u_t(0,x) = \varphi_0. \end{cases} \tag{4.3}$$

We denote $U_D(t)$ the linear wave group, defining the solution of (4.3)

$$\begin{array}{cccll} U_D(t): & H & \longrightarrow & H & \\ & (\varphi_0,\varphi_1) & \longmapsto & U_D(t)(\varphi_0,\varphi_1) & = (u(t), \partial_t u(t)) \end{array}$$

Let us consider the wave equation in exterior domain

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = 0 & \text{in } \mathbb{R}_+\times\Omega, \\ u = 0 & \text{on } \mathbb{R}_+\times\Gamma, \\ u(0,x) = \varphi_0 \quad \text{and} \quad \partial_t u(0,x) = \varphi_1, \end{cases} \tag{4.4}$$

where $(\varphi_0,\varphi_1) \in H$.

We denote $U_L(t)$ the linear wave group, defining the solution of (4.4)

$$\begin{array}{cccl} U_L(t): & H & \longrightarrow & H \\ & (\varphi_0,\varphi_1) & \longmapsto & U_L(t)(\varphi_0,\varphi_1) = (u(t),\partial_t u(t)) \end{array} \tag{4.5}$$

We choose $R>0$ such that B_R contains the obstacle O. Then we define spaces of outgoing and incoming data associated to solutions of problem (4.3) by

$$D_+^R = \{\varphi = (\varphi_0,\varphi_1)\in H \;;\; U_D(t)\varphi = 0 \text{ on } |x| < t+R,\ t\geq 0\}, \tag{4.6}$$

$$D_-^R = \{\varphi = (\varphi_0,\varphi_1)\in H \;;\; U_D(t)\varphi = 0 \text{ on } |x| < -t+R,\ t\leq 0\}. \tag{4.7}$$

These spaces satisfy the following properties:

(1) D_+^R and D_-^R are closed in H.
(2) D_+^R and D_-^R are orthogonal and

$$D_+^R \oplus D_-^R \oplus \left(\left(D_+^R\right)^\perp \cap \left(D_-^R\right)^\perp \right) = H. \tag{4.8}$$

Remark 4.1.

(1) *Solutions of (4.3) and (1.1) verify the finite speed propagation property.*
(2) *The nonlinearity being localized, it is easy to see that*

$$U(t) = U_D(t) \text{ on } D_+^R \text{ for every } t\geq 0. \tag{4.9}$$

(3) *Following* [14], *we denote by P_+ (resp.P_-) the orthogonal projection of H onto the orthogonal complement of D_+^R (resp.D_-^R). Thanks to (4.8), we easily deduce*

$$P_+\varphi \in \left(D_+^R\right)^\perp \cap \left(D_-^R\right)^\perp \text{ if } \varphi \in \left(D_-^R\right)^\perp. \tag{4.10}$$

(4) *The semi-group $U(t)$ operates on D_+^R for $t\geq 0$. Using the fact that the Cauchy problem admits a unique solution, we obtain:*

$$\begin{aligned} U(t)\varphi &= U(t)P_+\varphi + U(t)(I-P_+)\varphi \\ &= U(t)P_+\varphi + U_D(t)(I-P_+)\varphi, \end{aligned} \tag{4.11}$$

for every φ in H and for every $t\in\mathbb{R}_+$.

We denote by $K = \left(D_+^R\right)^\perp \cap \left(D_-^R\right)^\perp$, and we define the nonlinear Lax-Phillips operator on K by

$$Z(t) = P_+U(t)P_- \text{ for } t\geq 0. \tag{4.12}$$

In order to prove that $Z(t)$ operates on K, we need the following lemma.

Lemma 4.1.
Let $(\varphi,\psi)\in H\times H$ and $t\geq 0$, we have

$$\langle U(t)\varphi,\psi\rangle_H - \langle \varphi, U_D(-t)\psi\rangle_H = -\int_0^t \rho(s)\,\langle ag(\partial_t u(s)), \partial_t v(s-t)\rangle_{L^2(\Omega)}\,ds, \tag{4.13}$$

where we denoted by $U(t)\varphi = (u(t),\partial_t u(t))$ and $U_D(t)\psi = (v(t),\partial_t v(t))$.

Proof. Noting that for each $(u_0, u_1) \in H$ the solution u of (1.1) are given as a limit of smooth solution u_n with initial data $(u_{n,0}, u_{n,1})$ smooth such that $(u_{n,0}, u_{n,1}) \longrightarrow (u_0, u_1)$ in H. Note that $\|u_n(t,.) - u(t,.)\|_{H_D} + \|\partial_t u_n(t,.) - \partial_t u(t,.)\|_{L^2} \longrightarrow 0$, uniformly on $\mathbb{R}_+$. So we may assume that u is a smooth function. Let $\varphi \in H$ and $\psi \in H$. Thanks to Green formula

$$\begin{aligned} \frac{d}{dt}\langle U(t)\varphi, U_D(t)\psi\rangle_H &= \frac{d}{dt}\left(\langle \nabla u, \nabla v\rangle_{L^2} + \langle \partial_t u, \partial_t v\rangle_{L^2}\right) \\ &= \left\langle \partial_t^2 u - \Delta u, \partial_t v\right\rangle_{L^2} + \left\langle \partial_t u, \partial_t^2 v - \Delta v\right\rangle_{L^2} \\ &= \left\langle \partial_t^2 u - \Delta u, \partial_t v\right\rangle_{L^2} \end{aligned}$$

We easily deduce

$$\langle U(t)\varphi, U_D(t)\psi\rangle_H - \langle \varphi, \psi\rangle_H = -\int_0^t \rho(s)\,\langle ag(\partial_t u(s)), \partial_t v(s)\rangle_{L^2}\, ds.$$

Consequently, we obtain

$$\langle U(t)\varphi, \psi\rangle_H - \langle \varphi, U_D(-t)\psi\rangle_H = -\int_0^t \rho(s)\,\langle ag(\partial_t u(s)), \partial_t v(s-t)\rangle_{L^2}\, ds.$$

□

Using the result above we prove that the Lax-Phillips semi-group operates on K.

Proposition 4.1.
The semi-group $(Z(t))_{t\geq 0}$ operates on K.

Proof. Let $\varphi \in K$, and $t \geq 0$. According to (4.10), it suffices to verify that $U(t)\varphi \in \left(D_-^R\right)^{\perp}$. Let $(\varphi, \psi) \in \left(D_-^R\right)^{\perp} \times D_-^R$, then (4.13) yields,

$$\langle U(t)\varphi, \psi\rangle_H = \langle \varphi, U_D(-t)\psi\rangle_H - \int_0^t \rho(s)\,\langle ag(\partial_t u), \partial_t v(s-t)\rangle_{L^2(\Omega)}\, ds.$$

Knowing that $U_D(s-t)$ operates on D_-^R for $s \leq t$, and thus $U_D(s-t)\varphi = 0$ for $|x| \leq R + t - s$, we deduce that

$$\partial_t v(s-t)_{|B_R} = 0 \text{ on } [0,t].$$

This gives,

$$\langle U(t)\varphi, \psi\rangle_H = \langle \varphi, U_D(-t)\psi\rangle_H = 0.$$

□

5. Proof of the main Theorem

To prove the main Theorem we need some preliminary results.

5.1. **Preliminary results.** First we give the following result due to Aloui and Khenissi [1].

Proposition 5.1 ([1]). *We assume that (EGC) holds. Then, there exist $c_0 > 0$ and $T > 0$ such that*

$$\|Z_L(t)\|_{L(K)} = \|P_+ U_L(t) P_-\|_{L(K)} \le c_0 < 1, \tag{5.1}$$

for every $t \ge T$.

In the proposition below we prove a mixed observability.

Proposition 5.2. *There exist $T > 0$ and $C > 0$ such that for every φ in K , we have*

$$\begin{aligned} \|Z(t)\varphi\|_H^2 &\le C\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right) \\ &\quad + C\int_t^{t+T}\int_\Omega \rho(s)\, g(\partial_t u)\,\partial_t v(s-t) + \partial_t u \partial_t v(s-t)\, d\mathfrak{m}_a, \end{aligned} \tag{5.2}$$

for all $t \ge 0$, where u and v denote respectively the solution of (1.1) and (4.4) with initial data $\varphi = (\varphi_0, \varphi_1)$ and $((v_0, v_1) = Z(t)\varphi)$ in K.

Proof. Let $\varphi \in K$. According to Proposition 4.1, $Z(t)\varphi \in K$ for every $t \ge 0$. Setting $(\phi(s), \partial_s\phi(s)) = U_t(s) Z(t)\varphi$, where ϕ is the solution of

$$\begin{cases} \partial_s^2\phi - \Delta\phi + \rho(s+t)\, a(x)\, g(\partial_s\phi) = 0, & \mathbb{R}_+ \times \Omega, \\ \phi = 0, & \mathbb{R}_+ \times \Gamma, \\ (\phi(0), \partial_s\phi(0)) = Z(t)\varphi, \end{cases} \tag{5.3}$$

we have

$$U_t(s) Z(t)\varphi - U_D(s)(P_+ - I) U(t)\varphi = U(s+t)\varphi,$$

which implies in particular that

$$U_t(s) Z(t)\varphi = U(s+t)\varphi, \text{ on } B_R,$$

and

$$P_+ U_t(s) Z(t)\varphi = P_+ U(s+t)\varphi = Z(t+T)\varphi. \tag{5.4}$$

Then we obtain

$$\|Z(t+T)\varphi\|_H \le \|U_t(s) Z(t)\varphi - U_L(T) Z(t)\varphi\|_H + \|Z_L(T) Z(t)\varphi\|_H.$$

According to Proposition 5.1, there exist $0 < c_0 < 1$ and $T > 0$ such that,

$$\|Z(t+T)\varphi\|_H \le \|U_t(s) Z(t)\varphi - U_L(T) Z(t)\varphi\|_H + c_0 \|Z(t)\varphi\|_H. \tag{5.5}$$

On the other hand, let v solution of (4.4) with the same initial data $Z(t)\varphi$ in K. Then we define $z = \phi - v$, which satisfies the following system

$$\begin{cases} \partial_t^2 z - \Delta z + a(x)\,\rho(s+t)\, g(\partial_t\phi) - a(x)\partial_t v = 0, & \mathbb{R}_+ \times \Omega, \\ z = 0, & \mathbb{R}_+ \times \partial\Omega, \\ (z(0), \partial_t z(0)) = 0. \end{cases}$$

Since $Z(t)\varphi \in K$, then $a(x)\left(\rho(s+t)\,g(\partial_t\phi) - \partial_t v\right) \in L^2((0,T)\times\Omega)$. This observation permits us to apply energy identity, whence

$$E_z(T) = \int_{\Omega_T} a(x)\left(\partial_t v - \rho(s+t)\,g(\partial_t\phi)\right)\partial_t z\,dxdt \tag{5.6}$$

The monotonicity of g and $g(0)=0$ gives $g(s)s \geq 0$ for all $s \in \mathbb{R}$. Therefore using the identity (5.6), we get

$$\|U_t(s)\,Z(t)\,\varphi - U_L(T)Z(t)\,\varphi\|_H^2 \leq \int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a,$$

Using (5.5), we obtain

$$\begin{aligned}\|Z(t+T)\,\varphi\|_H &\leq \left(\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t u d\mathfrak{m}_a\right)^{1/2} + c_0\,\|Z(t)\,\varphi\|_H \\ &= \left(\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a\right)^{1/2} \\ &\quad + c_0\left(\|Z(t)\,\varphi\|_H - \|Z(t+T)\,\varphi\|_H\right) + c_0\,\|Z(t+T)\,\varphi\|_H\,.\end{aligned}$$

After some computation, we deduce that

$$\begin{aligned}\|Z(t+T)\,\varphi\|_H &\leq \frac{1}{1-c_0}\left(\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a\right)^{1/2} \\ &\quad + \frac{c_0}{1-c_0}\left(\|Z(t)\,\varphi\|_H - \|Z(t+T)\,\varphi\|_H\right),\end{aligned}$$

this gives

$$\begin{aligned}\|Z(t)\,\varphi\|_H &\leq \frac{1}{1-c_0}\left(\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a\right)^{1/2} \\ &\quad + \left(\frac{1}{1-c_0}\right)\left(\|Z(t)\,\varphi\|_H - \|Z(t+T)\,\varphi\|_H\right).\end{aligned}$$

Using the fact that for $a > b > 0$, $a - b \leq \left(a^2 - b^2\right)^{1/2}$, we obtain

$$\begin{aligned}\|Z(t)\,\varphi\|_H &\leq \tfrac{1}{1-c_0}\left(\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a\right)^{1/2} \\ &\quad + \left(\tfrac{1}{1-c_0}\right)\left(\|Z(t)\,\varphi\|_H^2 - \|Z(t+T)\,\varphi\|_H^2\right)^{1/2}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|Z(t)\,\varphi\|_H^2 \leq\ & C\int_0^T \int_\Omega \rho(s+t)\,g(\partial_t\phi)\,\partial_t v + \partial_t v\partial_t\phi d\mathfrak{m}_a \\ & + C\left(\|Z(t)\,\varphi\|_H^2 - \|Z(t+T)\,\varphi\|_H^2\right),\end{aligned}$$

with $C = \frac{2}{(1-c_0)^2}$.

Since

$$U_t(s)\,Z(t)\,\varphi = U(s+t)\,\varphi, \text{ on } B_R,$$

we obtain

$$\begin{aligned}\|Z(t)\varphi\|_H^2 \leq\ & C\int_0^T\int_\Omega \rho(s+t)\, g(\partial_t u(s+t))\,\partial_t v+\partial_t v\partial_t u(s+t)\, d\mathfrak{m}_a\\ & +C\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right).\end{aligned}$$

□

In the next we state some auxiliary results which will be used in the proof of Theorem 1. More precisely, arguing as in [9] we estimate the first term in the right hand side of the mixed observability estimate (5.2).

Lemma 5.1. *Let $t,T\geq 0$ and setting*

$$\Omega_t^0=\{(s,x)\in[t,t+T]\times\Omega;\ \ |\partial_s u(s,x)|<\eta_0\},\quad \Omega_t^1=\Omega_{t,t+T}\setminus\Omega_t^0.$$

We define

$$\Theta\left(\Omega_t^i\right)=\int_{\Omega_t^i}|\partial_s u(s)\,\partial_s v(s-t)|\, d\mathfrak{m}_a,\quad \Psi\left(\Omega_t^i\right)=\int_{\Omega_t^i}|g(\partial_s u(s))\,\partial_s v(s-t)|\, d\mathfrak{m}_a,\ \text{for } i=0,1,$$

where u and v denote respectively the solution of (1.1) and (4.4) with initial data (φ_0,φ_1) and $((v_0,v_1)=Z(t)\varphi)$ in H.

There exists a positive constant C which may depend on T such that the following inequalities hold for every $\epsilon>0$:

(1) *(Estimate on the damping near the origin)*

$$\begin{aligned}&\Theta(\Omega_t^0)+\Psi(\Omega_t^0)\leq\epsilon\|Z(t)\varphi\|_H^2\\ &+C\left(1+\tfrac{1}{\epsilon}\right)\mathfrak{m}_a(\Omega)\, h_0\left(\tfrac{1}{\mathfrak{m}_a(\Omega)}\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u\, d\mathfrak{m}_a\right).\end{aligned}\tag{5.7}$$

(2) *(Estimate on the damping near infinity)*

$$\Theta(\Omega_t^1)+\Psi(\Omega_t^1)\leq\epsilon\|Z(t)\varphi\|_H^2+C\epsilon^{-1}\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u\, d\mathfrak{m}_a.\tag{5.8}$$

Remark 5.1. *It is easy to see that*

$$\Theta\left(\Omega_t^0\right)+\Psi\left(\Omega_t^0\right)+\Theta\left(\Omega_t^1\right)+\Psi\left(\Omega_t^1\right)\leq\epsilon\|Z(t)\varphi\|_H^2+C\left(1+\frac{1}{\epsilon}\right)h\left(\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u d\mathfrak{m}_a\right),\tag{5.9}$$

for every $\epsilon>0$.

Proof. **Case (1).** Using Young's inequality, we obtain, for every $\epsilon>0$

$$\Theta(\Omega_t^0)+\Psi(\Omega_t^0)\leq\tfrac{1}{\epsilon}\int_{\Omega_t^0}\left[|\partial_t u|^2+|g(\partial_t u)|^2\right]d\mathfrak{m}_a+\epsilon\int_{\Omega_t^0}|\partial_t v(s-t)|^2\, d\mathfrak{m}_a.$$

The energy estimate yields

$$\Theta(\Omega_t^0)+\Psi(\Omega_t^0)\leq\tfrac{1}{\epsilon}\int_{\Omega_t^0}\left[|\partial_t u|^2+|g(\partial_t u)|^2\right]d\mathfrak{m}_a+\epsilon\|Z(t)\varphi\|_H^2$$

Using the inequality (1.3) for the function h_0 we obtain,

$$\int_{\Omega_t^0}\left[g(\partial_t u)^2+|\partial_t u|^2\right]d\mathfrak{m}_a\leq\frac{1}{\epsilon_0}\int_{\Omega_t^0}h_0\left(g(\partial_t u)\,\partial_t u\right)d\mathfrak{m}_a.$$

Now, since h_0 is concave, we use Jensen's inequality and we obtain

$$\begin{aligned}\int_{\Omega_t^0}\left(g\left(\partial_t u\right)^2+\left|\partial_t u\right|^2\right)d\mathfrak{m}_a &\leq \tfrac{T\mathfrak{m}_a(\Omega)}{\epsilon_0}h_0\left(\tfrac{1}{T\mathfrak{m}_a(\Omega)}\int_{\Omega_t^0}g\left(\partial_t u\right)\partial_t u\,d\mathfrak{m}_a\right)\\ &\leq \tfrac{T\mathfrak{m}_a(\Omega)}{\epsilon_0}h_0\left(\tfrac{1}{T\mathfrak{m}_a(\Omega)}\int_{\Omega_{t,t+T}}g\left(\partial_t u\right)\partial_t u\,d\mathfrak{m}_a\right).\end{aligned}$$

Finally, by combining the above estimates we deduce (5.7).

Case (2). Applying Young's inequality and using (1.4), we find

$$\Theta\left(\Omega_t^1\right)+\Psi\left(\Omega_t^1\right)\leq\epsilon\left\|Z\left(t\right)\varphi\right\|_H^2+C\epsilon^{-1}\int_{\Omega_{t,t+T}}g(\partial_s u)\partial_s u d\mathfrak{m}_a,$$

for every $\epsilon>0$. This concludes the proof of the lemma. □

Before giving the proof of Theorem 1, we give the following Lemma which is a time dependent version of the result in [12, lemma 3.3].

Lemma 5.2 ([7]). *Let*

- *$W(t)$ be a continuous, positive non-increasing function for $t\in\mathbb{R}_+$.*
- *θ is a non negative function $\mathbb{R}_+$. Let $T>0$ and setting, $\kappa(t)=T\sup\limits_{[t,t+T]}\theta(s)$, $t\geq 0$.*
- *Suppose for every $t\geq 0$, the functions $I-\kappa(t)\mathcal{L}(t,.):[0,W(0)]\to\mathbb{R}_+$ and $\mathcal{L}(t,.):[0,W(0)]\to\mathbb{R}_+$ are increasing, with $\mathcal{L}(t,0)=0$*
- *The function $\mathcal{L}(.,x):\mathbb{R}_+\to\mathbb{R}_+$ is decreasing for every x in $[0,W(0)]$.*
- *We assume that the following inequality*

$$W\left(\left(m+1\right)T\right)+\kappa\left(mT\right)\mathcal{L}\left(mT,W\left(mT\right)\right)\leq W\left(mT\right)\tag{5.10}$$

holds, for $m=0,1,2,\ldots$ Moreover, $\mathcal{L}(t,s)$ does not depend on m.

Then

$$W\left(t\right)\leq S\left(t-T\right),\qquad\forall t\geq T,$$

where $S(t)$ is the solution of the following nonlinear differential equation

$$\tfrac{dS}{dt}+\theta\left(t\right)\mathcal{L}\left(t,S\left(t\right)\right)=0;\qquad S(0)=W(0).\tag{5.11}$$

Moreover, if there exists $T_0>>1$ such that

$$\lim_{t\to+\infty}\int_{T_0}^t\theta\left(s\right)\mathcal{L}\left(s,\gamma\right)ds=+\infty,\tag{5.12}$$

for every $0<\gamma<<1$. Then

$$\lim_{t\to+\infty}S\left(t\right)=0.$$

For the proof of Lemma 5.2 we refer the reader to [7].

5.2. **Proof of Theorem 1.** Let $\varphi \in K$. For every $t \geq 0$, $Z(t)\varphi \in K$ and we have

$$\begin{aligned}
\|Z(t)\varphi\|_H^2 &\leq C\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right) \\
&\quad + C\int_t^{t+T}\int_\Omega \rho(s)\, g(\partial_t u)\,\partial_t v(s-t) + \partial_t u \partial_t v(s-t)\, d\mathfrak{m}_a \\
&\leq C\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right) \\
&\quad + C\int_t^{t+T}\int_\Omega \left[\rho(s)\, g(\partial_t u(s))\,\partial_s \tilde{v}(s) + \partial_t u(s)\,\partial_s \tilde{v}(s)\right] d\mathfrak{m}_a \qquad (5.13)
\end{aligned}$$

where $\tilde{v} = v(s-t)$, for $s \geq t$.

(1) **The function ρ is increasing on $\mathbb{R}_+$.**

$$\begin{aligned}
\|Z(t)\varphi\|_H^2 \leq\ & C\rho(t+T)\left[\Theta(\Omega_t^0) + \Psi(\Omega_t^0) + \Theta(\Omega_t^1) + \Psi(\Omega_t^1)\right] \\
& + C\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right) \\
=:\ & I_1 + I_2.
\end{aligned}$$

Using (5.13) and (5.9) and taking $\epsilon = \frac{1}{4\rho(t+T)}$ (where ϵ is the constant that appears in Lemma 5.1),we obtain

$$I_1 \leq C_T\left(\rho(t+T) + \rho^2(t+T)\right) h\left(\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u\; d\mathfrak{m}_a\right).$$

Since $\rho(t) \geq 1$ for all $t \geq 0$, we infer that

$$I_1 \leq \tfrac{C_T}{\alpha(t)} h\left(\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u\; d\mathfrak{m}_a\right),$$

where the function α is defined in Section 1. Now, using (5.3) and (5.4), we find

$$\begin{aligned}
\int_{\Omega_{t,t+T}} g(\partial_s u)\,\partial_s u d\mathfrak{m}_a &= \int_0^T\int_\Omega g(\partial_s \phi)\,\partial_s \phi d\mathfrak{m}_a \\
&\leq \frac{1}{\rho(t)}\int_0^T\int_\Omega \rho(s+t)\, g(\partial_s \phi)\,\partial_s \phi d\mathfrak{m}_a \\
&\leq \frac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2 - \|U_t(T)\, Z(t)\varphi\|_H^2\right) \\
&\leq \frac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right).
\end{aligned}$$

Thus

$$I_1 \leq \tfrac{C_T}{\alpha(t)} h\left(\tfrac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right)\right). \qquad (5.14)$$

For the term I_2 it is obvious that,

$$I_2 \leq \tfrac{C_T}{\alpha(t)}\left(\tfrac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2 - \|Z(t+T)\varphi\|_H^2\right)\right).$$

Combining the estimate above with (5.14), we obtain

$$\begin{aligned}\|Z(t)\varphi\|_H^2 &\leq \frac{C_T}{\alpha(t)}(I+h)\left[\frac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right)\right]\\ &\leq \frac{C_T}{\alpha(t)}h\left[\frac{1}{\rho(t)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right)\right].\end{aligned}$$

This yields,

$$\|Z(t+T)\varphi\|_H^2+\left(\sup_{[t,t+T]}T\beta(s)\right)h^{-1}\left(\frac{\alpha(t)\|Z(t)\varphi\|_H^2}{C_T}\right)\leq \|Z(t)\varphi\|_H^2.$$

(2) **The function ρ is decreasing on $\mathbb{R}_+$.**
We follows the same computations as in case 1. However, we take in this case $\epsilon=\frac{1}{4}$ and we obtain

$$I_1\leq C_T h\left(\int_{\Omega_{t,t+T}} g(\partial_s u)\partial_s u\, d\mathfrak{m}_a\right).$$

This implies,

$$I_1\leq \tfrac{C_T}{\alpha(t)}h\left(\textstyle\int_{\Omega_{t,t+T}} g(\partial_s u)\partial_s u\, d\mathfrak{m}_a\right)$$

where the function α is defined in Section 1. Now using the fact that ρ is decreasing, (5.3) and (5.4) we find

$$\begin{aligned}\int_{\Omega_{t,t+T}} g(\partial_s u)\partial_s u d\mathfrak{m}_a &= \int_0^T\int_\Omega g(\partial_s\phi)\partial_s\phi d\mathfrak{m}_a\\ &\leq \frac{1}{\rho(t+T)}\int_0^T\int_\Omega \rho(s+t)g(\partial_s\phi)\partial_s\phi d\mathfrak{m}_a\\ &\leq \frac{1}{\rho(t+T)}\left(\|Z(t)\varphi\|_H^2-\|U_t(T)Z(t)\varphi\|_H^2\right)\\ &\leq \frac{1}{\rho(t+T)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right).\end{aligned}$$

Therefore

$$I_1\leq \tfrac{C_T}{\alpha(t)}h\left(\tfrac{1}{\rho(t+T)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right)\right). \tag{5.15}$$

Next we estimate the term as follow, using $\rho(t)\leq 1$, for all $t\geq 0$,

$$I_2\leq \tfrac{C_T}{\alpha(t)}\left(\tfrac{1}{\rho(t+T)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right)\right).$$

Combining the estimate above with (5.15), we obtain

$$\|Z(t)\varphi\|_H^2\leq \frac{C_T}{\alpha(t)}h\left[\frac{1}{\rho(t+T)}\left(\|Z(t)\varphi\|_H^2-\|Z(t+T)\varphi\|_H^2\right)\right].$$

This gives,

$$\left\|Z\left(t+T\right)\varphi\right\|_H^2+\left(\sup_{[t,t+T]}T\beta\left(s\right)\right)h^{-1}\left(\frac{\alpha\left(t\right)\left\|Z\left(t\right)\varphi\right\|_H^2}{C_T}\right)\leq\left\|Z\left(t\right)\varphi\right\|_H^2,$$

for every $t\geq 0$, where the functions α and β are defined in Subsection 1.1. We summarize that in both cases ρ increasing and ρ decreasing we obtained that

$$\left\|Z\left(t+T\right)\varphi\right\|_H^2+\left(\sup_{[t,t+T]}T\beta\left(s\right)\right)h^{-1}\left(\frac{\alpha\left(t\right)\left\|Z\left(t\right)\varphi\right\|_H^2}{C_T}\right)\leq\left\|Z\left(t\right)\varphi\right\|_H^2, \tag{5.16}$$

for every $t\geq 0$ and for some $C_T>1$.

Let $K\geq C_T$ and consider $t=mT$,

$$\left\|Z\left(\left(m+1\right)T\right)\varphi\right\|_H^2+\left(\sup_{[mT,(m+1)T]}T\beta\left(s\right)\right)h^{-1}\left(\alpha\left(mT\right)\tfrac{\left\|Z(mT)\varphi\right\|_H^2}{K}\right)\leq\left\|Z\left(mT\right)\varphi\right\|_H^2$$

for all $m\in\mathbb{N}$.

Hence the above estimate and Lemma 5.2, imply the energy result in Theorem 1.

Now set $W\left(t\right)=E_u\left(t\right)$, $\theta\left(t\right)=\beta\left(t\right)$ and let

$$\mathcal{L}\left(t,x\right)=h^{-1}\left(\frac{\alpha\left(t\right)}{K}x\right),\ \text{ for }\ \left(t,x\right)\in\mathbb{R}_+\times\left[0,W\left(0\right)\right].$$

We recall that $\kappa\left(t\right)=T\sup\limits_{[t,t+T]}\theta\left(s\right)$. It is clear that, $0<\alpha\left(t\right)\leq 1$ and $0\leq\alpha\left(t\right)\kappa\left(t\right)\leq 1$, for every $t\geq 0$. Then for every $x_1>x_2$ in $\left[0,W\left(0\right)\right]$

$$\begin{aligned}&\left(I-\kappa\left(t\right)\mathcal{L}\right)\left(t,x_1\right)-\left(I-\kappa\left(t\right)\mathcal{L}\right)\left(t,x_2\right)\\&\geq\tfrac{x_1-x_2}{K}\left(K-\alpha\left(t\right)\kappa\left(t\right)\tfrac{h^{-1}((\alpha(t)x_1)/K)-h^{-1}((\alpha(t)x_2)/K)}{((x_1-x_2)\alpha(t))/K}\right),\end{aligned}$$

which yields,

$$\begin{aligned}&\left(I-\kappa\left(t\right)\mathcal{L}\right)\left(t,x_1\right)-\left(I-\kappa\left(t\right)\mathcal{L}\right)\left(t,x_2\right)\\&\geq\tfrac{x_1-x_2}{K}\left(K-\frac{h^{-1}\left(\left(\alpha\left(t\right)x_1\right)/K\right)-h^{-1}\left(\left(\alpha\left(t\right)x_2\right)/K\right)}{\left(\left(x_1-x_2\right)\alpha\left(t\right)\right)/K}\right).\end{aligned}$$

Using the fact that h^{-1} is positive and convex on $\left[0,W\left(0\right)\right]$, we obtain

$$\left|\tfrac{h^{-1}((\alpha(t)x_1)/K)-h^{-1}((\alpha(t)x_2)/K)}{((x_1-x_2)\alpha(t))/K}\right|\leq f_1\left(\left(\alpha\left(t\right)E_u(0)\right)/K\right),$$

where

$$\begin{array}{cccc}f_1: & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+\\ & x & \longmapsto & \frac{d^+}{dx}h^{-1}\left(x\right).\end{array}$$

Since f_1 is an increasing function on $\mathbb{R}_+$ and $\alpha\left(t\right)\leq 1$, we only have to choose K such that

$$K\geq f_1\left(E_u(0)/K\right)\geq f_1\left(\left(\alpha\left(t\right)E_u(0)\right)/K\right).$$

Knowing that $f_1\left(s\right)\leq 1$ for every s and $K>C_T>1$, we deduce that the function, $\left(I-\kappa\left(t\right)\mathcal{L}\left(t,.\right)\right)$ is increasing on $\left[0,W\left(0\right)\right]$, for every $t\geq 0$. Moreover it is clear that

the function $t\rightarrow\mathcal{L}(t,x)$ is decreasing on $\mathbb{R}_+$, for every x in $[0,W(0)]$ and $\mathcal{L}(t,.):[0,W(0)]\rightarrow\mathbb{R}_+$ is increasing for every $t\geq 0$. Now, from lemma 5.2, we infer that

$$E_u(t)\leq S(t-T), \qquad \text{for all } t\geq T,$$

where $S(t)$ is the solution of the following nonlinear differential equation

$$\frac{dS}{dt}+\beta(t)\,h^{-1}\left(\frac{\alpha(t)}{K}S(t)\right)=0, \qquad S(0)=E_u(0).$$

Furthermore, if for some $T_0\gg 1$, the following result

$$\int_{T_0}^{t}\beta(s)\,h^{-1}\left(\tfrac{\gamma\alpha(s)}{K}\right)ds\underset{t\rightarrow+\infty}{\longrightarrow}+\infty,$$

holds, for every $0<\gamma\ll 1$. We conclude that

$$\|Z(t)\varphi\|_H\underset{t\rightarrow+\infty}{\longrightarrow}0.$$

This completes the proof of Theorem 1.

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A. BCHATNIA, Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunisia and University of DAMMAM, KSA
E-mail address: ahmed.bchatnia@fst.rnu.tn

M. DAOULATLI, Department of Mathematics, Faculty of Sciences of Bizerte, University of Carthage, Tunisia
E-mail address: moez.daoulatli@infcom.rnu.tn